Solving infinitary Rubik's cubes

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The conundrum

Being somewhat informal for the moment, consider a "$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ Rubik's cube", an infinitary analogue of the $N \times N \times N$ Rubik's cube.

As with an ordinary Rubik's cube, we imagine scrambling it by twisting various layers. We then endeavour to return it to the initial (solved) state through a further sequence of twists.

It seems sensible to apply an infinite sequence of twists and end up with a perfectly good configuration (at least sometimes. . .)

No reason to stop there, the process is inherently transfinite.

The conundrum

On an finite Rubik's cube, we know every attainable configuration can be solved.

Just execute the scramble in reverse!

This argument fails on its face for transfinite sequences of twists.

Question

On an infinite Rubik's cube, can every attainable configuration (perhaps via a transfinite sequence of twists) be solved?

Main results

Two variations, the *edged* and *edgeless* cubes of arbitrary cardinalities $\aleph_\alpha$

Theorem (T.)

For the edged cube of cardinality $\aleph_\alpha$, if σ is a (transfinite) sequence of twists which converges when applies to the solved state, then there is sequence τ (of length $< \omega_{\alpha+1}$) which inverts σ.

Theorem (T.)

For the countable edgeless cube, if f is a configuration obtained from the solved state by a convergent (transfinite) sequence, then f is solvable in at most ω^2 many moves. I.e., there is a sequence of twists of length $\leq \omega^2$ which yields the solved configuration when applied to f.

The edgeless cube $\mathcal{Q}_L$

- Fix an infinite set L. Let $-L$ be a copy of L (writing $-r \in -L$ for the copy of each $r \in L$) and let $0, \pm\infty$ denote distinguished objects not in $\pm L$.
- For convenience, define the order $-\infty < -r < 0 < r < +\infty$ for $r \in L$.
- Write $L^\dagger = -L \cup \{0\} \cup L$ and $\bar{L}^\dagger = [-\infty, +\infty] = L^\dagger \cup \{\pm\infty\}$.
- One should think of the structure $\mathcal{Q}_L$ we are defining as the "$L^\dagger \times L^\dagger \times L^\dagger$" Rubik's cube:

Definition (The edgless cube $\mathcal{Q}_L$)

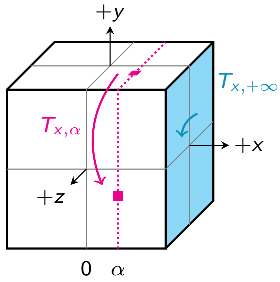
Let $U = \bar{L}^\dagger \times \bar{L}^\dagger \times \bar{L}^\dagger$ endowed with (x, y, z) coordinates and let $\mathcal{Q}_L$ be the (disjoint union of) six copies of $L^\dagger \times L^\dagger$ embedded in the extreme $\pm\infty$ planes of U in the obvious way. Points of $\mathcal{Q}_L$ (i.e., points of U with exactly one $\pm\infty$ coordinate) are called *cells*.

Twists

Definition

A *quarter-turn twist* $T_{i,\alpha} : \mathcal{Q}_L \to \mathcal{Q}_L$ for $i \in \{x, y, z\}$ and $\alpha \in \bar{L}^\dagger$ is permutation of $\mathcal{Q}_L$ given by a rotation of the cells in the $i = \alpha$ plane according to the right-hand rule.

- E.g., $T_{x,\alpha}(\alpha, +\infty, \beta) = (\alpha, -\beta, +\infty)$ for $\alpha \in L^\dagger$. (Right, in magenta.)
- $T_{i,\pm\infty}$ are *face twists*. For example, $T_{x,+\infty}(+\infty, \beta, \gamma) = (+\infty, -\gamma, \beta)$ is a rotation of the right face $\{x = +\infty\}$. (Right, in cyan.)
- NB, face twists non-trivially permute *only* the cells in the concerned face. This is really what we mean by "edgeless".



Configurations

Definition

- A *configuration* of $\mathcal{Q}_L$ is a map $f : \mathcal{Q}_L \to \Gamma \cup \{\varnothing\}$ to the gamut $\Gamma = \{\mathtt{r}, \mathtt{b}, \mathtt{w}, \mathtt{o}, \mathtt{g}, \mathtt{y}\}$ together with the special value $\varnothing$.
- A configuration is *legal* if it does not take the value $\varnothing$.
- The *solved configuration* is the assignment

$$f_{\text{solved}}(x, y, z) = \begin{cases} \mathtt{r} & \text{if } x = +\infty, \\ \mathtt{b} & \text{if } y = +\infty, \\ \mathtt{w} & \text{if } z = +\infty, \\ \mathtt{o} & \text{if } x = -\infty, \\ \mathtt{g} & \text{if } y = -\infty, \\ \mathtt{y} & \text{if } z = -\infty. \end{cases}$$

Action of twist sequences

- Quarter-, half-, and reverse quarter-turn twists $T_{i,\alpha}$, $T_{i,\alpha}^2$, $T_{i,\alpha}^3$ are called *basic twists*. A *basic sequence* is a sequence $\langle \sigma_\eta : \eta < \theta \rangle$ of basic twists σ_η for some ordinal θ.
- Basic twists act on configurations in the natural way, namely, if T is a basic twist and f a configuration, then the configuration Tf is given by $Tf(c) = f(T^{-1}c)$.
- Given a basic sequence $\langle \sigma_\eta : \eta < \theta \rangle$ and an initial configuration f_0, we define the corresponding sequence of configurations $\langle f_\eta : \eta \leq \theta \rangle = \langle \sigma_\eta : \eta < \theta \rangle * f_0$ as follows.
  - At successor stages, we act by the next element of the twist sequence, $f_{\eta+1} = \sigma_\eta f_\eta$.
  - At limit stages λ, we set $f_\lambda(c) = \lim_{\eta \nearrow \lambda} f_\eta(c)$ if this limit exists and $f_\lambda(c) = \varnothing$ otherwise.
- Note that this always yields a terminal configuration f_θ (even if θ is a limit ordinal). We write $f_\theta = \langle \sigma_\eta : \eta < \theta \rangle \cdot f_0$.

Convergence

Definition

Consider a basic sequence $\sigma = \langle \sigma_\eta : \eta < \theta \rangle$ and the corresponding sequence $\langle f_\eta : \eta \leq \theta \rangle = \sigma * f_0$ for a (legal) initial configuration f_0.

We say that the sequence $\langle f_\eta : \eta \leq \theta \rangle$ is convergent if f_θ is legal. In this case, we say that the twist sequence σ is convergent over f_0.

Variants

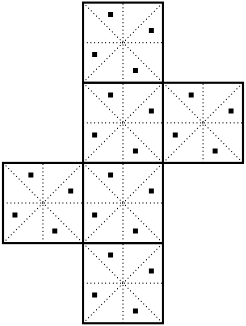
- The *edged* cube $\bar{\mathcal{Q}}_L$ is defined similarly except for the following.
  - Its faces are copies of $\bar{L}^\dagger \times \bar{L}^\dagger$ (rather than $L^\dagger \times L^\dagger$).
  - Strictly speaking, $\bar{\mathcal{Q}}_L$ is not embedded in U since more than one cell (along edges of adjacent faces) may have the same position. In such ambiguous cases, we specify a cell by underlining the coordinate specifying the face, e.g., $(\alpha, -\infty, \underline{+\infty})$ is the Front cell at the Front Down edge in the $x = \alpha$ layer.
  - For $\bar{\mathcal{Q}}_L$, face twists also affect the adjacent edge and corner cells in the obvious way. For example, $T_{y,-\infty}(\alpha, -\infty, \underline{+\infty}) = (\underline{+\infty}, -\infty, -\alpha)$.
- We may also consider (egded and edgeless) cubes analogous to $N \times N \times N$ for *even* N by omitting (or ignoring) the 0 layers.

Cell clusters

Definition (Cell cluster)

The cluster of a cell c is its orbit under the basic twists.

- A cell/cluster is *center* if two of its coords are 0, *cross* if one coord is 0, *diagonal* if its face coords are equal or opposite, *edge* if it has two $\pm\infty$ coords, or *corner* if it has three $\pm\infty$ coords.
- Clusters partition the cube. Excepting the center cluster, every cluster has exactly 24 cells. (The center cluster has 6.)



Observation

Illegality is persistent in the sense that no legal configuration can be obtained from an illegal one (in any ordinal number of moves).

Proof.

Suppose f_0 is illegal, i.e., $f_0(c_0) = \varnothing$ for some c_0.
Apply a basic sequence $\langle \sigma_\eta : \eta < \theta \rangle$ and suppose that every configuration f_η except possibly f_θ has a colorless cell in the cluster C of c_0, i.e., $f_\eta(c_\eta) = \varnothing$ for some $c_\eta \in C$ for each $\eta < \theta$.
If $\theta = \xi + 1$ is a successor ordinal, then
$f_\theta(\sigma_\xi c_\xi) = f_{\xi+1}(\sigma_\xi c_\xi) = f_\xi(c_\xi) = \varnothing$.
If θ is a limit, by finiteness of C, c_η has a cofinal constant subsequence $c_{\eta_\nu} = c$ and then $f_\theta(c) = \varnothing$.
Thus $f_\theta(c) = \varnothing$ for some $c \in C$ and the claim follows by induction. □

Definition (Equivalence of twist sequences)

Two basic sequences σ, τ are equivalent if $\sigma \cdot f_0 = \tau \cdot f_0$ for every initial configuration f_0. In this case, we write $\sigma \sim \tau$.

The relevant algebraic structure is the monoid of basic sequences under concatenation modulo $\sim$.

Twist-finite sequences

Twist-finiteness

Say a basic sequence σ is twist-finite if each basic twist appears in it only finitely many times.

- Since each cell is affected non-trivially by only nine basic twists, it follows that every twist-finite sequence is convergent over every (legal) initial configuration.
- The twist-finite sequences are closed under $\sim$, hence form a submonoid.
- In fact, they form a group in which every element has finite order dividing $K = \operatorname{lcm}\{|\pi| : \pi \in S_{24}\}$.
  - *Proof sketch.* Every twist-finite sequence σ acts as a product of permutations of clusters $\prod_C \pi_C$. Finite iterates σ^n are also twist-finite. σ^K acts via $(\prod_C \pi_C)^K = \prod_C \pi_C^K = \mathrm{id}$. Therefore $[\sigma^{K-1}] = [\sigma]^{-1}$.

Edged cubes

A priori, twist-finiteness is too restrictive, we really only care about convergence over f_{solved}.

These notions are the same for the edged cube $\bar{\mathcal{Q}}_L$!

Say distinct edge/corner cells are *coupled* if they have the same coordinates, e.g., $(\alpha, \underline{+\infty}, +\infty)$ and $(\alpha, +\infty, \underline{+\infty})$.

Lemma

If f is a legal configuration accessible from f_{solved}, then for each pair $\gamma, \gamma' \in \Gamma$ of colors and each non-corner edge cluster C, there is at most one pair of coupled edge cells c, c' with $c \in C$ and $f(c) = \gamma$ and $f(c') = \gamma'$. Similarly, for each $\gamma, \gamma', \gamma'' \in \Gamma$, there is at most one triple of coupled corner cells c, c', c'' such that $f(c) = \gamma$, $f(c') = \gamma'$, $f(c'') = \gamma''$.

Proof.

Let σ convergent over f_{solved} and consider $\sigma * f_{\text{solved}}$.
We argue no stage can first violate the lemma.
f_{solved} has the property. Successor case is trivial. If a violation occurs at a limit stage, then the (two or three) witnessing cells must have stabilized color by some earlier stage. □

Theorem

For the edged cube $\bar{\mathcal{Q}}_L$, then sequences convergent over the solved configuration are exactly the twist-finite sequences.

Proof.

Clearly the twist-finite sequences converge over f_{solved}.
Conversely, suppose $\langle \sigma_\eta : \eta < \theta \rangle$ is convergent over f_{solved} but some twist T appears along an infinite subsequence $\sigma_{\eta_\nu} = T$.
WLOG, θ is a limit ordinal and $\theta = \sup_\nu \eta_\nu$.
First, suppose T is not a face twist and let c, c' be any pair of coupled edge cells affected non-trivially by T.
Then,
$f_\theta(Tc) = \lim_\nu f_{\eta_\nu+1}(Tc) = \lim_\nu Tf_{\eta_\nu}(Tc) = \lim_\nu f_{\eta_\nu}(c) = f_\theta(c)$
and likewise $f_\theta(Tc') = f_\theta(c')$. But Tc and Tc' are also coupled, contradiction.
If T is a face twist, argue similarly with a triple of coupled corner cells affected by T. □

Every accessible legal configuration is solvable!

The main theorem for the edged cube follows immediately.

Corollary

Every legal configuration of the edged cube $\bar{\mathcal{Q}}_{\aleph_\alpha}$ accessible from f_{solved} is solvable (in $< \omega_{\alpha+1}$ many moves).

Open question: Is $< \omega_{\alpha+1}$ optimal?

We have answered the question of *solvability in principle* (for the edged cube) but the result is not fully satisfactory.

- The solution sequence depends not only on the scrambled configuration f, but on a sequence of twists σ obtaining f.

Perhaps we can adapt finite Rubik's cube algorithms.

- For the edged cube, I do not yet know how to do so in a convergent manner.

Countable edgeless cube

- For the $N \times N \times N$ cube, each cluster can be solved individually in $O(1)$ many moves while leaving all other clusters unchanged.
- Thus, solving all clusters in series yields an $O(N^2)$ solution. Naïvely adapting to the infinite case typically diverges.
- By cleverly parallelizing the cluster solutions, one can improve the overall solution to $O(N^2/\log N)$ (which is in fact optimal).
- We can do something similar in the countable edgeless case to obtain an ω^2 solution procedure which works for a broad class of configurations including all accessible ones (and possibly more).

Standard configurations

Definition

A configuration of the edgeless cube $\mathcal{Q}_L$ is *standard* if the center cluster is in the solved configuration up to a global rotation and every non-center cluster has exactly four of each color of Γ.

- (By induction) every legal configuration accessible from f_{solved} is standard.
- Standardness is clearly a necessary condition for solvability.
- Every standard non-center cluster configuration is WLOG an *even* permutation of the solved configuration. (Otherwise, transpose two like-colored cells.)

Face quadrants and cluster configurations

- Let $\mathcal{D} = L \times (\{0\} \cup L) \times \{+\infty\}$ be the Upper Right quadrant of the Front face. We identify $\mathcal{D}$ with its xy projection.
- The action of the basic twists partitions $\mathcal{Q}_L$ into 24 translates of $\mathcal{D}$ and the center cluster.
- We may specify a (non-center) cluster, denoted $C(\alpha, \beta)$ by its representative $(\alpha, \beta) \in \mathcal{D}$. In particular, the diagonal clusters are the $C(\alpha, \alpha)$ and the cross clusters are the $C(\alpha, 0)$.
- Each face quadrant contains exactly one cell of each non-center cluster and this gives a 1-1 correspondence between clusters.
- We will say that two clusters *have the same configuration* if corresponding cells have the same color.

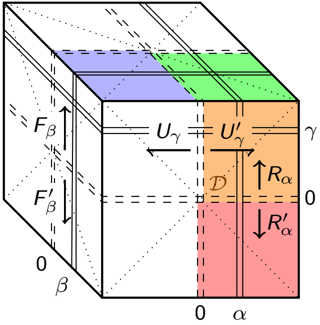
Individual cluster solutions

By case analysis, the sequence

$$\langle R_\alpha, F'_\beta, R'_\alpha, F_\beta, U'_{+\infty}, F_\beta, R_\alpha, F'_\beta, R'_\alpha, U_{+\infty}\rangle \qquad (*)$$

in the notation of the diagram cycles the three cells of the (β, α) cluster among the quadrants highlighted in red, greed, and blue and does not affect any other cells.

- Other cells may change temporarily but are restored.
- (This is a *commutator* in the Rubik's jargon.)
- Only involves slice twists in $\pm\alpha, \pm\beta$ layers and face twists.



Individual cluster solutions

For each choice of front and top face, we get such a sequence which yields a 3-cycle on the (β, α) cluster.

These 24 cycles generate the even permutations on the cluster.

Assembling finitely many such sequences, we can solve any give cluster $C(\beta, \alpha)$ using only $\pm\alpha, \pm\beta$ slice twist and face twists.

Solving clusters in parallel

If many clusters are in the same configuration and have a certain product structure, we can *parallelize* these solutions so that they share the same face twists.

Lemma

Suppose $X \times Y \subset \mathcal{D}$ *with* X *and* Y *disjoint. If all clusters* $X \times Y$ *have the same configuration, then there is a twist-finite basic sequence of ordinal length* $\leq (|X| + |Y| + 1) \cdot k$ *for some finite* k *involving only face moves and slice moves with index in* $X \cup Y$ *which, when applied to the given configuration solves all clusters* $X \times Y$ *and fixes (i.e., restores) all other clusters except possibly* $(X \times Y) \cup (X \times X) \cup (Y \times Y)$.

Proof.

Let h be the common configuration of the $X \times Y$ clusters.
These clusters admit cluster solutions with common type sequences $a_1, \ldots, a_k$, $b_1, \ldots, b_k$, and $c_1, \ldots, c_k$ (depending only on h).
For each $1 \leq i \leq k$, let $s_i = \langle S_{a_i,x} : x \in X \rangle ^\frown \langle S_{b_i,y} : y \in Y \rangle ^\frown \langle F_{c_i} \rangle$.
Slice moves of the same type commute so we may take s_i to have length $|X| + |Y| + 1$ (possibly shorter due to identity moves).
Consider the twist-finite sequence $s = s_1 {}^\frown s_2 {}^\frown \cdots {}^\frown s_k$.
Any cell in cluster $C(\alpha, \beta)$ can only possibly be affected by face moves and slice moves of index α or β. Thus, the cluster of each $(x, y) \in X \times Y$ is affected only by the subsequence $\langle S_{a_1,x}, S_{b_1,y}, F_{c_1}, \ldots, S_{a_k,x}, S_{b_k,y}, F_{c_k} \rangle$ of s which is a cluster solution for h. (NB, this relies on the disjointness of X and Y.)
Similarly, the subsequence which affects each other cluster has no effect by the properties of cluster solutions, with the possible exceptions of $X \times X$ and $Y \times Y$. □

So far, the results of this section apply to the edgeless cube of arbitrary cardinality. Now we restrict our attention to the countable case.

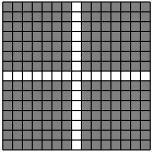
Theorem

On the countable edgeless cube $\mathcal{Q}_L$ with $|L| = \aleph_0$, every standard configuration is solvable in at most ω^2 many moves.

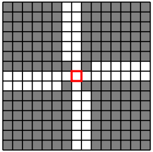
Proof.

Let $L = \{1, 2, 3, \dots\}$ endowed with the usual order.
The solution proceeds in ω many stages.
As a preliminary step, solve the center cluster $C(0, 0)$.
There are at most $m = |A_{24}| = 24!/2$ many cluster configurations.
Stage 0: For each of the m cluster configurations h in turn, let X_h be the set of $x \in L$ where $C(x, 0)$ is in configuration h and solve $X_h \times \{0\}$ in $\leq (|X_h| + 2) \cdot k \leq \omega \cdot k + 2$ many moves.
Stage n: For each h in turn, let X_h be the set of $x \in L \setminus \{0, 1, \dots, n\}$ with $C(x, n)$ in configuration h and solve all $X_h \times \{n\}$. Similarly, solve each $\{n\} \times Y_h$ in turn. Finally, solve the diagonal cluster $C(n, n)$ (in finitely many moves). Altogether, this stage uses $\leq (\omega \cdot km + 2) \cdot 2 + \omega \leq \omega \cdot (km + 1)$ many moves.
Stage n uses only face twists and slice twists of index $\geq n$, so each cluster is affected only by face twists after some stage.
f_{solved} is invariant under face twists, so every cluster stabilizes on f_{solved} before stage ω^2. □

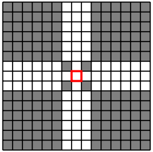
End stage 0



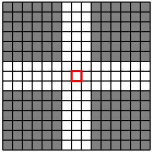
Stage 1a



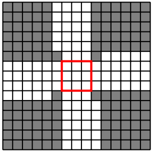
Stage 1b



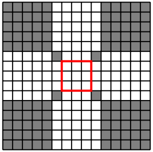
Stage 1c



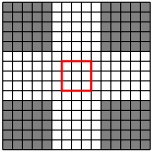
Stage 2c



Stage 2b



Stage 2b



- This argument breaks down in the uncountable case. The obstruction is the Cartesian product structure imposed by the parallelization lemma. In the countable case, we can afford to consider only one row/column at a time, thus trivializing the product structure.
- This solution also fails on the (countable) edged cube because it uses infinitely many face twists. In fact, when applied to the edged cube, it converges to f_{solved} except along the edges.
- But we can push the argument a little further:

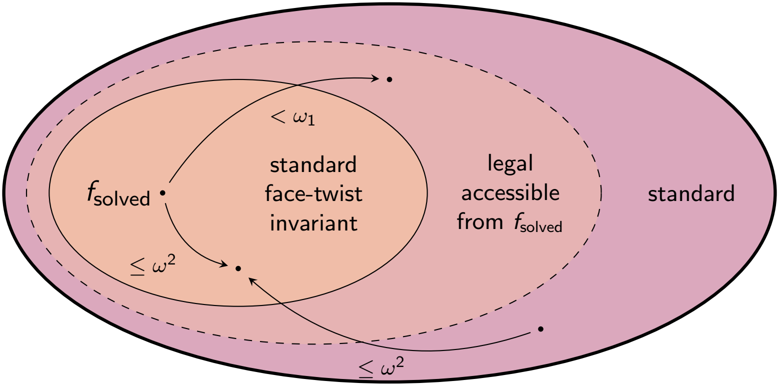
Theorem

For the countable edgeless cube, every standard configuration which is invariant under face twists can be obtained from any standard configuration in $\leq \omega^2$ many moves.

Proof sketch.

Parallelize cluster solutions by desired permutation. □

Accessibility from and to f_{solved}



Open questions

- Are all accessible configurations of *uncountable* edgeless cubes solvable? All standard configurations?
- Are there any standard but inaccessible configurations?
- Can the $\leq \omega^2$ solution length for $\mathcal{Q}_{\aleph_0}$ be improved? Can it be improved uniformly?
- In the edgeless case, are there configurations only accessible via non-twist-finite sequences?
- Do the answers to the above differ if we do not allow half-turn twists?
- Is there a uniform (fast) solution procedure for the edged cube?